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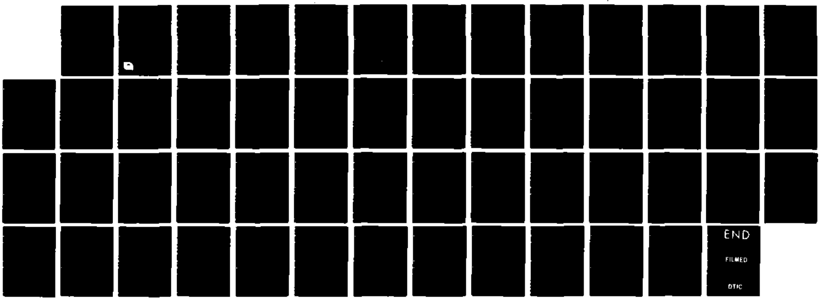

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ROBUST PREALLOCATED PREFERENTIAL DEFENSE

Jerome Bracken
Peter S. Brooks
James E. Falk

August 1985

IDA

INSTITUTE FOR DEFENSE ANALYSES
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The problem is to protect a set of T identical targets that may come under attack by A identical weapons. The targets are to be defended by D identical interceptors, which must be preallocated to defend selected targets. The attacker is aware of the number of interceptors, but is ignorant of their allocation.
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The size of the attack is chosen by the attacker from within a specified range. The robust strategies developed in this paper do not require the defender to assume an attack size. Rather, the defender chooses a strategy which is good over a wide range of attack sizes, though not necessarily best for any particular attack size. The attacker, knowing that the defender is adopting a robust strategy, chooses the optimal attack strategy for the number of weapons he chooses to expend. The expected number of survivors is a function of the robust defense strategy and optimal attack strategy against this robust defense.

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ROBUST PREALLOCATED PREFERENTIAL DEFENSE

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I. INTRODUCTION

The problem that we address is the protection of a set of T identical targets that may come under attack by A identical attacking weapons. The targets are to be defended by D identical defensive interceptors which must be preallocated to defend selected targets. The attacker is aware of the number of interceptors but is ignorant of their allocation.

In two seminal papers ([6] and [7]; see also [8]) Matheson addressed the case where the defender knows the size A of the potential attack but not its allocation. He represented the scenario as a two-person, zero-sum game by allowing the attacker and defender to choose allocations x and y independently, and adopted the expected fraction of surviving targets as the payoff function. We refer to this as the basic game.

Later ([3],[4],[5],[9] and [10]) a number of authors developed allocation procedures based on linear programming solution procedures for solving the game-theoretic problem studied by Matheson. Current modelling is being performed which utilizes these methods. In all cases, the attack size A is assumed to be known.

Figure 1 exhibits the value of the basic game as a function of attack size for a typical Matheson game with 1000 targets and 6000 defenders. Each point on the curve labeled "game value" represents the proportion of targets surviving an attack of A weapons when the attacker knows that D=6000 and the defender knows A, but neither knows how these weapons are to be deployed. With both the attacker and defender selecting optimal strategies in the sense of game theory, the game value plotted in Figure 1 represents the outcome of these strategies.

A basic assumption implicit in the Matheson game is that both "players" are acting as though they are playing the same game, i.e., they are both informed of all of the parameters and

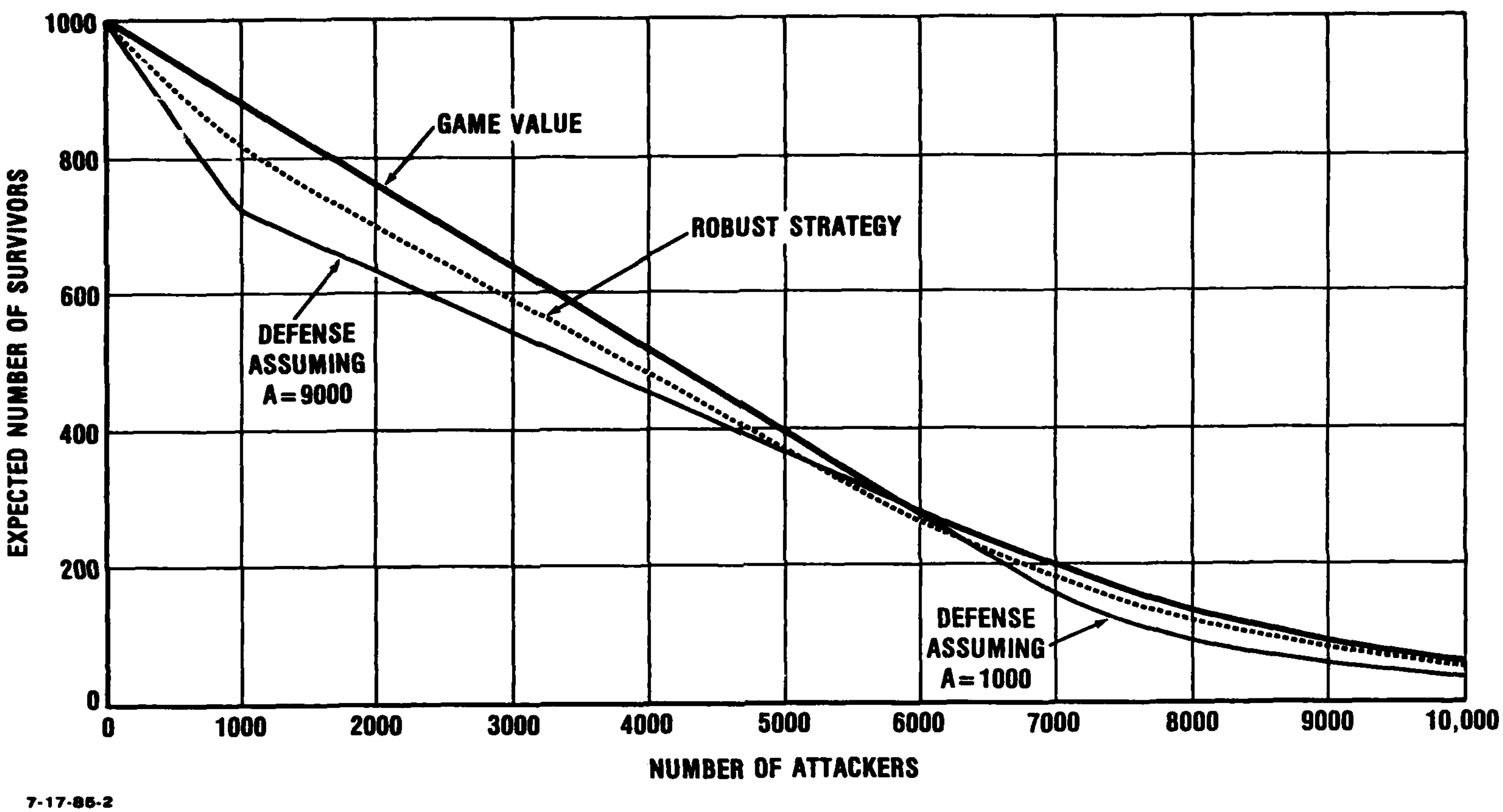

Figure 1. RESULTS FOR ALTERNATIVE SITUATIONS

rules of the game. For each attack size A, there is a specific game, all other parameters remaining the same.

However, the actual attack size is an option of the attacker, who can choose to attack with any number of weapons up to his total inventory. If he is interested only in minimizing the proportion of surviving targets, the monotonically decreasing nature of the game value curve will impel him to attack with his total inventory. If, however, the attacker has other interests and attacks with fewer than his total inventory, or if the defender overestimates the attacker's maximum inventory, or if a weaker opponent attacks, other results can be expected.

Suppose, for example, that the defender assumes in his planning an attack of 9000. Knowing this, the attacker actually attacks with 1000. If the defender were to use a strategy optimal against an optimal attack of 1000 (i.e., if the defender knew which game the attacker had chosen) he could expect about 880 survivors. However, if the attacker were to discover the defender's assumption, he could take advantage of it by optimizing against it. Doing so, he could bring the expected number of surviving targets down to about 720 (see Figure 1).

Alternatively, assume that the defender assumes an attack size of 1000. The attacker, knowing this, attacks with 9000. Had the defender planned on the basis of 9000, the expected number of survivors would be about 85, but the mis-planning of the defense would yield an expected number of survivors of about 57 (see Figure 1).

The robust strategies developed in this paper do not require the defender to assume an attack size. Rather, the defender chooses a strategy which is good over a wide range of attack sizes, though not necessarily best for any particular attack size. The attacker, knowing that the defender is adopting a robust strategy, chooses the optimal attack strategy for the number of weapons he chooses to expend, and the expected number of survivors is based on this attack and allocation.

Figure 1 shows the game value, the results of the defense assuming two attack sizes and the attacker taking advantage of this, and the results of a robust strategy. In the above example, the robust defense yields expected survivors of 820 of 880 (as compared with 720 of 880) when the attack is 1000, and 79 of 85 (as compared with 57 of 85) when the attack is 9000.

In the main body of this paper we study the expected number of survivors under two behavioral assumptions for the defender and two for the attacker, resulting in four separate cases. The defender may (a) believe the attacker will use the optimal strategy of the basic game or (b) believe the attacker will use a strategy optimal against the defender's robust strategy. The attacker may (a) use the optimal strategy of the basic game or (b) use a strategy optimal against the defender's robust strategy. Thus, including the basic game, we examine five separate cases and show that a fairly wide range of outcomes results from the various assumptions.

In Section II, we summarize the basic game of Matheson and its equivalent linear program. Section III introduces the notion of a robust strategy and defines the behavioral assumptions for the defender and the attacker. Section IV contains the mathematical problems addressed in the various cases. In Section V we present an example solved for four combinations of kill parameters for the defense and the offense. The appendices discuss alternative physical assumptions on the engagement at each target.

II. THE BASIC GAME

In this section we summarize the basic game to be discussed and set down the underlying assumptions and notation. The summary is based on Matheson ([6] and [7]) and Hogg [4].

There are:

- T targets of equal value to be defended
- A missiles attacking these targets
- D defending interceptors.

Integer values of T, A and D are given. We consider T = 1000, A = 1000, 2000,...,10,000, and D = 6000 in the examples of this paper.

Also given for each attack and defense allocation is a value of:

p_{ij} = probability that a target under attack by i attacking missiles and j defending interceptors will survive.

The particular values of p_{ij} result from specific assumptions on how the attack and defense at each target interact. In the original work of Matheson ([6] and [7]), it was assumed that the attack could be sequentially numbered and that at most one defender could engage an attacker. In our paper, the numerical work displayed in the main body is based on the assumption of a <u>simultaneous</u> attack repelled by a "uniform defense" at each target. In Appendix A we show that this is optimal for the defender when the attack size is known and derive an explicit expression for p_{ij}. In Appendix B we present results for the assumption of <u>sequential attack of unknown size</u>. The Prim-Read firing doctrine is utilized by the defense. Comparisons are made of results for simultaneous and sequential attacks. In Appendix C we address the case of <u>sequential attack of known size</u>, and with a defender "shoot-look-shoot" capability, giving recursion relations for computing p_{ij} for various attacks and defenses.

We assume that (1) at most R attacking missiles can attack a single target and (2) at most S defending interceptors can defend a single target. In this paper we set $R = S = 10$.

The attacker and defender must choose an allocation of their inventories to the set of targets. Both A and D are known beforehand, as well as the p_{ij}'s, but the actual allocations are unknown to the opponents.

If we set

x_i = fraction of the targets to be attacked by i attacking missiles

and

y_j = fraction of the targets to be defended by j defending interceptors,

then it has been shown (in the above references) that

$$S(x,y) = x^T P y$$

gives the fraction of the T targets that are expected to survive under attack and defense strategies x and y, where

$$x = (x_0, \ldots , x_R),$$
$$y = (y_0, \ldots , y_S)$$

and

$$P = (p_{ij}).$$

The value of the basic game G is:

$$v^*(A) = \underset{\substack{\sum_{j=0}^{S} y_j = 1 \\ \sum_{j=1}^{S} j y_j = D/T \\ y_j \geq 0}}{\text{maximum}} \quad \underset{\substack{\sum_{i=0}^{R} x_i = 1 \\ \sum_{i=1}^{R} i x_i = A/T \\ x_i \geq 0}}{\text{minimum}} \quad x^T P y$$

BASIC GAME SOLUTION

A	$y^*(A)$											$x^*(A)$											$v^*(A)$
1,000	.115	.070	.077	.069	.044	.006	.053	.080	.110	.145	.230	.833	.014	.014	.015	.015	.016	.015	.018	.017	.019	.026	.914
2,000	.115	.070	.077	.069	.044	.006	.053	.080	.110	.145	.230	.665	.028	.027	.030	.029	.032	.031	.035	.033	.037	.052	.828
3,000	.115	.070	.077	.069	.044	.006	.053	.080	.110	.145	.230	.498	.042	.041	.045	.044	.049	.046	.053	.050	.056	.078	.742
4,000	.115	.070	.077	.069	.044	.006	.053	.080	.110	.145	.230	.331	.056	.054	.060	.058	.065	.061	.070	.066	.074	.104	.656
5,000	.115	.070	.077	.069	.044	.006	.053	.080	.110	.145	.230	.163	.070	.068	.075	.073	.081	.077	.088	.083	.093	.130	.570
6,000	.133	.061	.075	.069	.046	•	.048	.074	.103	.136	.254	•	.083	.080	.089	.094	.066	.118	.104	.098	.111	.156	.485
7,000	.359	•	.039	.011	•	•	•	•	•	.021	.570	•	•	.143	.106	•	•	•	•	.483	.132	.134	.425
8,000	.368	•	.038	•	•	•	•	•	•	.016	.578	•	•	.155	.004	•	•	•	•	•	.736	.105	.377
9,000	.400	•	•	•	•	•	•	•	•	.001	.599	•	•	.010	•	•	•	•	•	•	.920	.070	.333
10,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	•	•	1.000	.290

DEFENDER CASE I: $y^{I} = (0.4, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, 0.6)$

A	$v^*_{I,\,I}(A)$	$x(y^{I})$											$v^*_{I,\,II}(A)$
1,000	.914	•	1.000	•	•	•	•	•	•	•	•	•	.720
2,000	.828	•	•	1.000	•	•	•	•	•	•	•	•	.636
3,000	.742	•	•	.857	•	•	•	•	•	•	.143	•	.593
4,000	.656	•	•	.714	•	•	•	•	•	•	.286	•	.550
5,000	.570	•	•	.571	•	•	•	•	•	•	.429	•	.506
6,000	.485	•	•	.429	•	•	•	•	•	•	.571	•	.463
7,000	.425	•	•	.286	•	•	•	•	•	•	.714	•	.419
8,000	.377	•	•	.143	•	•	•	•	•	•	.857	•	.377
9,000	.333	•	•	•	•	•	•	•	•	•	1.000	•	.333
10,000	.290	•	•	•	•	•	•	•	•	•	•	1.000	.290

DEFENDER CASE II: $y^{II} = (0.303, 0, 0.079, 0.049, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, 0.570)$

A	$v^*_{II,\,I}(A)$	$x(y^{II})$											$v^*_{II,\,II}(A)$
1,000	.914	•	1.000	•	•	•	•	•	•	•	•	•	.788
2,000	.828	•	•	1.000	•	•	•	•	•	•	•	•	.710
3,000	.742	•	•	•	1.000	•	•	•	•	•	•	•	.657
4,000	.656	•	•	•	•	1.000	•	•	•	•	•	•	.581
5,000	.570	•	•	•	•	.667	•	•	.333	•	•	•	.527
6,000	.485	•	•	•	•	.333	•	•	.666	•	•	•	.472
7,000	.425	•	•	•	•	•	•	•	1.000	•	•	•	.418
8,000	.375	•	•	•	•	•	•	•	•	1.000	•	•	.364
9,000	.318	•	•	•	•	•	•	•	•	•	1.000	•	.317
10,000	.276	•	•	•	•	•	•	•	•	•	•	1.000	.276

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Table 1. VALUES AND STRATEGIES FOR a,d = .7, .7

BASIC GAME SOLUTION

A	y*(A)											x*(A)											v*(A)
1,000	.131	.146	•	•	•	•	.345	•	•	•	.379	.814	.017	.008	•	•	•	.161	•	•	•	•	.878
2,000	.131	.146	•	•	•	•	.345	•	•	•	.379	.628	.034	.016	•	•	•	.322	•	•	•	•	.755
3,000	.131	.146	•	•	•	•	.345	•	•	•	.379	.442	.051	.024	•	•	•	.483	•	•	•	•	.633
4,000	.131	.146	•	•	•	•	.345	•	•	•	.379	.255	.068	.032	•	•	•	.645	•	•	•	•	.511
5,000	.131	.146	•	•	•	•	.345	•	•	•	.379	.069	.084	.041	•	•	•	.806	•	•	•	•	.389
6,000	.340	.034	•	•	•	•	•	.098	•	•	.528	•	.064	.074	•	•	•	.248	.615	•	•	•	.283
7,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	1.000	•	•	•	.192
8,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	1.000	•	•	.128
9,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	•	1.000	•	.085
10,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	•	•	1.000	.057

DEFENDER CASE I: $y^{I}=(0.4, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, 0.6)$

A	$v^*_{I,I}(A)$	$x(y^I)$											$v^*_{I,II}(A)$
1,000	.878	•	1.000	•	•	•	•	•	•	•	•	•	.720
2,000	.755	•	.833	•	•	•	•	•	.166	•	•	•	.632
3,000	.633	•	.667	•	•	•	•	•	.333	•	•	•	.544
4,000	.511	•	.500	•	•	•	•	•	.500	•	•	•	.456
5,000	.389	•	.533	•	•	•	•	•	.667	•	•	•	368
6,000	.283	•	.167	•	•	•	•	•	.833	•	•	•	.280
7,000	.192	•	•	•	•	•	•	•	1.000	•	•	•	.192
8,000	.128	•	•	•	•	•	•	•	•	1.000	•	•	.128
9,000	.085	•	•	•	•	•	•	•	•	•	1.000	•	.085
10,000	.057	•	•	•	•	•	•	•	•	•	•	1.000	.057

DEFENDER CASE II: $y^{II}=(0.247, 0.007, 0.149, 0.040, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, 0.558)$

A	$v^*_{II,I}(A)$	$x(y^{II})$											$v^*_{II,II}(A)$
1,000	.877	•	1.000	•	•	•	•	•	•	•	•	•	.815
2,000	.754	•	.500	•	.500	•	•	•	•	•	•	•	.702
3,000	.630	•	•	•	1.000	•	•	•	•	•	•	•	.588
4,000	.508	•	•	•	.667	•	•	.333	•	•	•	•	.482
5,000	.384	•	•	•	.333	•	•	.667	•	•	•	•	.376
6,000	.281	•	•	•	•	•	•	1.000	•	•	•	•	.270
7,000	.179	•	•	•	•	•	•	•	1.000	•	•	•	.179
8,000	.119	•	•	•	•	•	•	•	•	1.000	•	•	.119
9,000	.079	•	•	•	•	•	•	•	•	•	1.000	•	.079
10,000	.053	•	•	•	•	•	•	•	•	•	•	1.000	.053

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.7, .7
.7, .9
.9, .7
.9, .9

For each of these a,d pairs are computed, and displayed in Tables 1-4:

(a) The basic game value $v^*(A) = S(x^*(A), y^*(A))$ and corresponding optimal strategies $x^*(A)$ and $y^*(A)$ for $\bar{A} = \{1000, 2000, \ldots, 10{,}000\}$.

(b) The defender's robust strategy y^I under the defender's assumption that the attacker will use his optimal strategies $x^*(A)$ based on the game value.

(i) The expected fraction of survivors $v^*_{I,I}(A) = S(x^*(A), y^I)$ when the defender is correct,

(ii) The optimal attacker response $x(y^I)$ and corresponding value $v^*_{I,II}(A) = (S(x(y^I), y^I)$ when the defender is mistaken.

(c) The defender's robust strategy y^{II} under the defender's assumption that the attacker will base his attack on y^{II}.

(i) The expected fraction $v^*_{II,I}(A) = S(x^*(A), y^{II})$ when the defender is mistaken,

(ii) The optimal attacker response $x(y^{II})$ and corresponding value $v^*_{II,II}(A) = S(x(y^{II}), y^{II})$ when the defender is correct.

V. EXAMPLES

We present four examples to illustrate the concepts discussed in Section III. In each of these examples we consider

$$T = 1000$$
$$D = 6000$$
$$\overline{A} = \left\{1000,\ 2000,\ \ldots,\ 10{,}000\right\}$$

with

$$R = S = 10 \quad .$$

The survival probabilities p_{ij} are computed by the formula

$$p_{ij} = \left(1-a(1-d)^{[j/i] + 1}\right)^{i\langle j/i\rangle}\left(1-a(1-d)^{[j/i]}\right)^{i(1-\langle j/i\rangle)}$$

where

a = probability that a single attacker will destroy an undefended target.

d = probability that a single interceptor will destroy an attacker at which it is directed.

The basic underlying assumption here is that the defense will spread its defenders as uniformly as possible over the attackers. This assumption is justified in Appendix A, and it is shown to result in a defense which maximizes the probability that the target will survive.

The four examples contained herein differ only in the values a and d, which are set equal to the a,d pairs:

$\underline{v^*_{II,I}(A)}$

In this case, the defender again solves the above problem (LP(II,II)) to obtain an optimal solution y^{II}. Here, however, the attacker is using a strategy which is optimal for each attack size A. Thus the value is

$$v^*_{II,I}(A) = x^*(A)^T P y^{II} \qquad (A \in \bar{A}).$$

subject to:

$$\begin{aligned} s(A) &\le P_0 y \\ s(A) - t(A) &\le P_1 y \\ &\vdots \\ s(A) - Rt(A) &\le P_R y \end{aligned}$$

where $y \in Y$ is given. Thus the defense sees the problem as

$$\underset{y \in Y}{\text{maximize}} \quad \underset{A \in \bar{A}}{\text{minimum}} \left\{ (1/v^*(A)) \underset{s(A),t(A)}{\text{maximum}} \left\{ s(A) - (A/T)t(A) \right\} \right\}$$

where $s(A)$ and $t(A)$ are restricted as above. But this is equivalent to choosing y, ρ, $s(A)$ for $A \in \bar{A}$ and $t(A)$ for $A \in \bar{A}$ which solve:

$$\left.\begin{array}{l} \underset{y \in Y}{\text{maximize}}\ \rho \\ \\ \text{subject to:} \\ \left.\begin{aligned} \rho v^*(A) &\le s(A) - (A/T)t(A) \\ s(A) &\le P_0 y \\ s(A) - t(A) &\le P_1 y \\ &\vdots \\ s(A) - Rt(A) &\le P_R y \end{aligned}\right\} A \in \bar{A} \end{array}\right\} \text{LP(II,II)}$$

If we denote the solution of this linear program by y^{II} and the optimal responses (obtained from (1) above) by $x(y^{II})$ $(A \in \bar{A})$ we have

$$v^*_{II,II}(A) = x(y^{II})^T P y^{II} \qquad (A \in \bar{A}).$$

If the solution of this problem is y^I, then

$$v^*_{I,I}(A) = x^*(A)^T P y^I \qquad (A \in \bar{A})$$

where $x^*(A)$ solves LP(0).

<u>$v^*_{I,II}(A)$</u>

In this case the defender sees the same problem as above and constructs the same robust strategy y^I. Now, however, the attacker knows y^I and hence can optimize against it.
Thus

$$v^*_{I,II}(A) = \underset{x \in X(A)}{\text{minimum}} \; x^T P y^I \qquad (A \in \bar{A})$$

where y^I solves LP(I,I) and where X(A) is defined analogous to Y.

<u>$v^*_{II,II}(A)$</u>

In this case the defender still wants to solve the problem

$$\underset{y \in Y}{\text{maximize}} \quad \underset{A \in \bar{A}}{\text{minimum}} \quad \left\{ R_A(x,y) \right\}$$

but now, however,

$$R_A(x,y) = (1/v^*(A)) \underset{x \in X(A)}{\text{minimum}} \; x^T P y \qquad (1)$$

This is a linear program (for each fixed y) and its dual is

$$(1/v^*(A)) \quad \text{maximum} \quad \left\{ s(A) - (A/T)t(A) \right\}$$

IV. LINEAR PROGRAMMING EQUIVALENTS

Each of the problems described above can be formulated and hence solved as a linear program. In this section we present the problems addressed.

In Section II, we derived the game value $v^*(A)$ in terms of the optimal value of a linear program LP(0). The solution of such linear programs (one for each value of A) has been denoted by $(x^*(A), y^*(A))$. We now wish to derive expressions for the expected fraction of surviving targets under the various assumptions.

$v^*_{I,I}(A)$

By definition, the defense seeks to solve the problem

$$\underset{y \in Y}{\text{maximize}} \quad \underset{A \in \overline{A}}{\text{minimum}} \quad x^*(A)^T P y / v^*(A)$$

But this is equivalent to the linear program

$$\left.\begin{array}{l} \text{maximize } \rho \\ \text{subject to:} \\ \quad y \in Y \\ \quad \rho \geq 0 \\ \quad \rho \leq x^*(A)^T P y / v^*(A) \text{ for all } A \in \overline{A} \end{array}\right\} \quad LP(I,I)$$

ATTACKER RESPONSE I: The attacker employs his optimal game strategy $x^*(A)$ for any $A \in \bar{A}$.

ATTACKER RESPONSE II: The attacker is capable of recognizing and optimally adapting to any strategy that the defender employs.

We could, alternatively, view this pair of assumptions as distinguishing the correctness of the defender's assumptions. Keeping in mind that the defender is, in all cases, interested in building a robust strategy, we summarize the four possible combinations as follows:

I,I (Defender I, Attacker I)	The defender sees the attacker as "uninformed". The defender is correct. The attacker is "uninformed".
I,II (Defender I, Attacker II)	The defender sees the attacker as "uninformed". The defender is mistaken. The attacker is "informed".
II,I (Defender II, Attacker I)	The defender sees the attacker as "informed". The defender is incorrect. The attacker is "uninformed".
II,II (Defender II, Attacker II)	The defender sees the attacker as "informed". The defender is correct. The attacker is "informed".

of expected surviving value to game value. But he is now faced with the question of defining the expected surviving value for each attack size. It turns out that this definition is critically dependent on the defender's assumptions of the attacker's behavior, and the correctness of this assumption.

We distinguish two "defender assumptions".

DEFENDER ASSUMPTION I: The attacker will employ his optimal game strategy $x^*(A)$ for any particular attack size A.

DEFENDER ASSUMPTION II: The attacker can discover, and therefore optimize against, whatever strategy the defender employs.

Thus, with assumption I, the defender feels that the attacker is oblivious to the defender's desire to install a robust defense. In game-theoretic terms, the defender assumes that the attacker, in spite of the fact that he will choose the game (i.e., choose A), views the defender as playing the game optimally, and hence will use his own optimal strategy $x^*(A)$.

Such an assumption is, of course, vulnerable to exploitation. Either player in a two-person, zero-sum game can expect the game value if he employs <u>any</u> combination of his active strategies, <u>as long as his opponent uses his optimal game strategy</u>. However, a player who deviates from his optimal game strategy is vulnerable, and can generally not expect the game value if his opponent learns of the deviation.

With assumption II, we give the defender the foresight or good sense to recognize his opponent's capability to predict and to take advantage of a particular defense strategy.

In order to determine the actual expected outcomes $R_A(x,y^*)$ for a given y^*, we further distinguish a pair of alternatives reflecting the correctness of the defender's assumptions.

Thus we seek to determine:

$$\underset{y \in Y}{\text{maximum}} \quad \underset{A \in \overline{A}}{\text{minimum}} \quad \left\{R_A(x,y)\right\}$$

where

$$Y = \left\{(y_0,\ldots,y_S): \quad \sum_{j=0}^{S} y_j = 1, \quad \sum_{j=1}^{S} j y_j = D/T, \; y_j \geq 0\right\}$$

$$\overline{A} = \left\{1000, \; 2000,\ldots,10{,}000\right\}$$

As it stands, the problem is not well-defined because we have not specified the vector x. We do this by making assumptions on the ways that the defender views his opponent and on the correctness of these views.

We adopt the ratio measure because we wish to do relatively well in all cases. In particular, we wish to avoid the situations discussed in the introductory section. We do not wish to plan for a small attack and fail almost completely if the attack is large, for we are interested in preserving some missiles for finite deterrence. Also, we do not wish to plan for a large attack and lose a substantial portion of our force if the attack is small, in order to deny any potential aggressor an attractive small exchange of his missiles for ours. Thus, we are concerned about the behavior of the entire range of the expected survivors as a function of attack size. We choose a ratio measure rather than a difference measure because this problem is in the context of many other strategic nuclear weapon planning problems, and we cannot visualize how a difference measure across a wide range of attacking weapons would fit into the overall planning context. We are satisfied with performing relatively well over the range of interest.

The defender desires to choose a strategy y* which is "robust against attack size", i.e., maximizes the smallest ratio

III. STRATEGIES ROBUST AGAINST ATTACK SIZE

For each possible attack size A, there is an associated game value $v^*(A)$, as well as a pair of (generally mixed) optimal strategies $(x^*(A), y^*(A))$. However, we would not expect the attacker to divulge the specific value of A before the attack begins. Since $y^*(A)$ is optimal for the defender only over an interval $[A^-, A^+]$, it generally will not represent an optimal strategy for values outside of this interval. We are therefore led to look for a single strategy which will be "robust" over a range of A values so broad that no single defense $y^*(A)$ is optimal for the basic game described in the previous section.

Obviously, from an attacker's point of view, if he wished to minimize the total expected fraction of targets surviving, he would attack with his largest arsenal because $v^*(A)$ is a non-increasing function of A. A defender who knew that the attacker would use an entire arsenal A would prepare for an attack of that size and hence use the strategy $y^*(A)$.

If, however, A were not known, or known only to be an upper bound on the attacker's arsenal, and if the defense had to be concerned with smaller attacks, $y^*(A)$ would generally be sub-optimal.

We assume that there is given a value A' which represents an <u>upper bound</u> on the attack size. In this paper we set A' = 10,000.

As a measure of robustness, we choose the ratio defined by the expected fraction of surviving targets $S_A(x,y)$ for an attack of size A divided by the optimal game value $v^*(A)$:

$$R_A(x,y) = S_A(x,y)/v^*(A)$$

and seek to determine a defense strategy y^* which maximizes the smallest of these ratios over a set of A values.

we obtain

$$s = \left(\frac{1}{j-i}\right) (jP_i - iP_j) \cdot y$$

$$t = \left(\frac{1}{j-i}\right) (P_i - P_j) \cdot y \qquad .$$

If we assume that

$$p_{ik} \geq p_{jk}$$

for each k (more attacking weapons on a target will decrease the probability of survival for a fixed number of defenders), then

$$P_i - P_j \geq 0$$

so that we may assume $t \geq 0$, and therefore $s \geq 0$, i.e., non-negativity conditions may be imposed on all variables of the above linear program. Non-negativity is convenient for computational reasons.

Let $x^*(A)$, $y^*(A)$ and $v^*(A)$ denote the solution of the above linear program. Note that the dual variables associated with the inequality constraints of LP(0) constitute $x^*(A)$.

Note that LP(0), as a function of the parameter A (which occurs in the objective function only), defines the value $v^*(A)$ for each $A \geq 0$. It is well-known that $v^*(A)$ is piecewise linear and convex. It is also clearly monotonically nonincreasing.

This value can be used to measure the effectiveness of a defense of size D against an attack of size A, where the probabilities p_{ij} are given.

By taking the dual of the inside problems defining G, we obtain the linear program:

$$
\left.\begin{array}{l}
v^*(A) = \text{maximum } [s - (A/T)t] \\
\text{subject to:} \\
\begin{aligned}
s \quad &\leq P_0 y \\
s - t &\leq P_1 y \\
&\ \vdots \\
s - Rt &\leq P_R y \\
\sum_{j=0}^{S} y_j &= 1 \\
\sum_{j=1}^{S} j y_j &= D/T \\
y_j &\geq 0
\end{aligned}
\end{array}\right\} \text{LP}(0)
$$

where P_i denotes the ith row of P.

Note that, for any feasible y, at least two of the inequality constraints above must be binding at an (s,t) pair which solves LP(0). Let $i < j$ denote the indices of two such constraints. From

$$
\begin{aligned}
s - it &= P_i y \\
s - jt &= P_j y
\end{aligned}
$$

Table 3. VALUES AND STRATEGIES FOR a,d = .9, .7

BASIC GAME SOLUTION

A	y*(A)											x*(A)											v*(A)
1,000	.101	.162	•	•	•	•	.361	•	.044	•	.332	.797	.016	.016	•	.038	•	.133	•	•	•	•	.865
2,000	.101	.162	•	•	•	•	.361	•	.044	•	.332	.593	.032	.032	•	.076	•	.267	•	•	•	•	.730
3,000	.101	.162	•	•	•	•	.361	•	.044	•	.322	.390	.048	.049	•	.113	•	.400	•	•	•	•	.596
4,000	.101	.162	•	•	•	•	.361	•	.044	•	.332	.186	.064	.065	•	.151	•	.534	•	•	•	•	.461
5,000	.165	.069	.139	•	•	•	.151	•	•	•	.474	•	.076	.082	.010	.132	•	.701	•	•	•	•	.327
6,000	.311	.092	.007	•	•	•	•	•	•	•	.589	•	.052	.058	•	•	•	.400	.491	•	•	•	.224
7,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	1.000	•	•	•	.132
8,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	1.000	•	•	.077
9,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	•	1.000	•	.044
10,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	•	•	1.000	.026

DEFENDER CASE I: $y^{I} = (0.4, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, 0.6)$

A	$v^*_{I,I}(A)$	$x(y^I)$											$v^*_{I,II}(A)$
1,000	.865	•	1.000	•	•	•	•	•	•	•	•	•	.640
2,000	.730	•	.833	•	•	•	•	•	.167	•	•	•	.555
3,000	.596	•	.667	•	•	•	•	•	.333	•	•	•	.471
4,000	.461	•	.500	•	•	•	•	•	.500	•	•	•	.386
5,000	.327	•	.333	•	•	•	•	•	.667	•	•	•	.301
6,000	.224	•	.167	•	•	•	•	•	.833	•	•	•	.217
7,000	.132	•	•	•	•	•	•	•	1.000	•	•	•	.132
8,000	.077	•	•	•	•	•	•	•	•	1.000	•	•	.077
9,000	.044	•	•	•	•	•	•	•	•	•	1.000	•	.044
10,000	.026	•	•	•	•	•	•	•	•	•	•	1.000	.026

DEFENDER CASE II: $y^{II} = (0.202, 0.039, 0.164, 0.044, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, 0.550)$

A	$v^*_{II,I}(A)$	$x(y^{II})$											$v^*_{II,II}(A)$
1,000	.864	1.000	•	•	•	•	•	•	•	•	•	•	.793
2,000	.723	•	.500	•	.500	•	•	•	•	•	•	•	.669
3,000	.591	•	•	•	1.000	•	•	•	•	•	•	•	.546
4,000	.458	•	•	•	.667	•	•	.333	•	•	•	•	.434
5,000	.326	•	•	•	.333	•	•	.667	•	•	•	•	.321
6,000	.223	•	•	•	•	•	•	1.000	•	•	•	•	.209
7,000	.121	•	•	•	•	•	•	•	1.000	•	•	•	.121
8,000	.078	•	•	•	•	•	•	•	•	1.000	•	•	.078
9,000	.041	•	•	•	•	•	•	•	•	•	1.000	•	.041
10,000	.024	•	•	•	•	•	•	•	•	•	•	1.000	.024

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Table 4. VALUES AND STRATEGIES FOR a,d = .9, .9

BASIC GAME SOLUTION

A	y*(A)											x*(A)											v*(A)
1,000	.081	.072	.063	.078	.039	.064	.046	.067	.135	.192	.166	.830	.014	.014	.015	.015	.017	.016	.019	.018	.020	.022	.907
2,000	.081	.072	.063	.078	.039	.064	.046	.067	.135	.192	.166	.661	.028	.028	.031	.030	.034	.033	.038	.035	.040	.043	.815
3,000	.081	.072	.063	.078	.039	.064	.046	.067	.135	.192	.166	.491	.042	.041	.046	.046	.051	.049	.056	.053	.060	.066	.722
4,000	.081	.072	.063	.078	.039	.064	.046	.067	.135	.192	.166	.321	.056	.055	.061	.061	.068	.065	.075	.071	.080	.087	.629
5,000	.081	.072	.063	.078	.039	.064	.046	.067	.135	.192	.166	.152	.070	.069	.077	.076	.085	.082	.093	.088	.100	.108	.537
6,000	.203	.041	.066	.072	.027	•	•	.032	.067	.107	.377	•	.081	.080	.089	.097	•	.189	.119	.103	.117	.126	.445
7,000	.329	.016	.064	.003	•	•	•	•	•	.028	.559	•	.069	.068	.084	•	•	•	•	.557	.130	.091	.380
8,000	.332	.016	.064	•	•	•	•	•	•	.027	.562	•	.059	.063	.025	•	•	•	•	•	.785	.067	.326
9,000	.392	.009	•	•	•	•	•	•	•	•	.599	•	.056	•	•	•	•	•	•	•	.493	.450	.279
10,000	.400	•	•	•	•	•	•	•	•	•	.600	•	•	•	•	•	•	•	•	•	•	1.000	.234

DEFENDER CASE I: $y^I = (0.400, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, 0.600)$

A	$v^*_{I,I}(A)$	$x(y^I)$											$v^*_{I,II}(A)$
1,000	.907	•	1.000	•	•	•	•	•	•	•	•	•	.640
2,000	.815	•	.889	•	•	•	•	•	•	•	•	.111	.595
3,000	.722	•	.778	•	•	•	•	•	•	•	•	.222	.550
4,000	.629	•	.667	•	•	•	•	•	•	•	•	.333	.505
5,000	.537	•	.556	•	•	•	•	•	•	•	•	.444	.459
6,000	.445	•	.444	•	•	•	•	•	•	•	•	.556	.414
7,000	.380	•	.333	•	•	•	•	•	•	•	•	.667	.369
8,000	.326	•	.222	•	•	•	•	•	•	•	•	.778	.324
9,000	.279	•	.111	•	•	•	•	•	•	•	•	.889	.279
10,000	.234	•	•	•	•	•	•	•	•	•	•	1.000	.234

DEFFNDER CASE II: $y^{II}=(0.187, 0.059, 0.107, 0.078, 0.034, \bullet, \bullet, \bullet, \bullet, \bullet, 0.536)$

A	$v^*_{II,I}(A)$	$x(y^{II})$											$v^*_{II,II}(A)$
1,000	.907	•	1.000	•	•	•	•	•	•	•	•	•	.825
2,000	.815	•	.500	•	.500	•	•	•	•	•	•	•	.728
3,000	.722	•	•	•	1.000	•	•	•	•	•	•	•	.638
4,000	.629	•	•	•	•	1.000	•	•	•	•	•	•	.555
5,000	.537	•	•	•	•	.667	•	•	.333	•	•	•	.489
6,000	.445	•	•	•	•	.333	•	•	.667	•	•	•	.424
7,000	.378	•	•	•	•	•	•	•	1.000	•	•	•	.358
8,000	.321	•	•	•	•	•	•	•	•	1.000	•	•	.299
9,000	.264	•	•	•	•	•	•	•	•	•	1.000	•	.258
10,000	.209	•	•	•	•	•	•	•	•	•	•	1.000	.209

12-12-84-7

Figures 2-5 display the expected number of survivors as a function of the number of attackers for the cases studied.

The expected number of survivors always equals the game value in I,I. This is so because the active strategies (those pure strategies corresponding to positive components of a mixed strategy) of y^{I} are always among the active strategies of $y^*(A)$. It is well-known in game theory that one player in a two-person zero-sum game can play his active strategies with <u>any</u> probability distribution and receive the game value provided that his opponent is playing his optimal strategy. Also, the robust strategy y^{II} yields expected numbers of survivors that are nearly identical to the game value when the attacker is uninformed of the defender's strategy.

The worst case for the defender is situation I,II, where the attacker takes advantage of the defender's incorrect assumption about the attacker's behavior. Note that the outcome for the smaller attacks differs substantially from the game value while the outcome for the larger attacks is the same.

By contrast, situation II,II lies below the game value for all attacks, being superior to I,II for smaller attacks and slightly inferior to I,II for larger attacks.

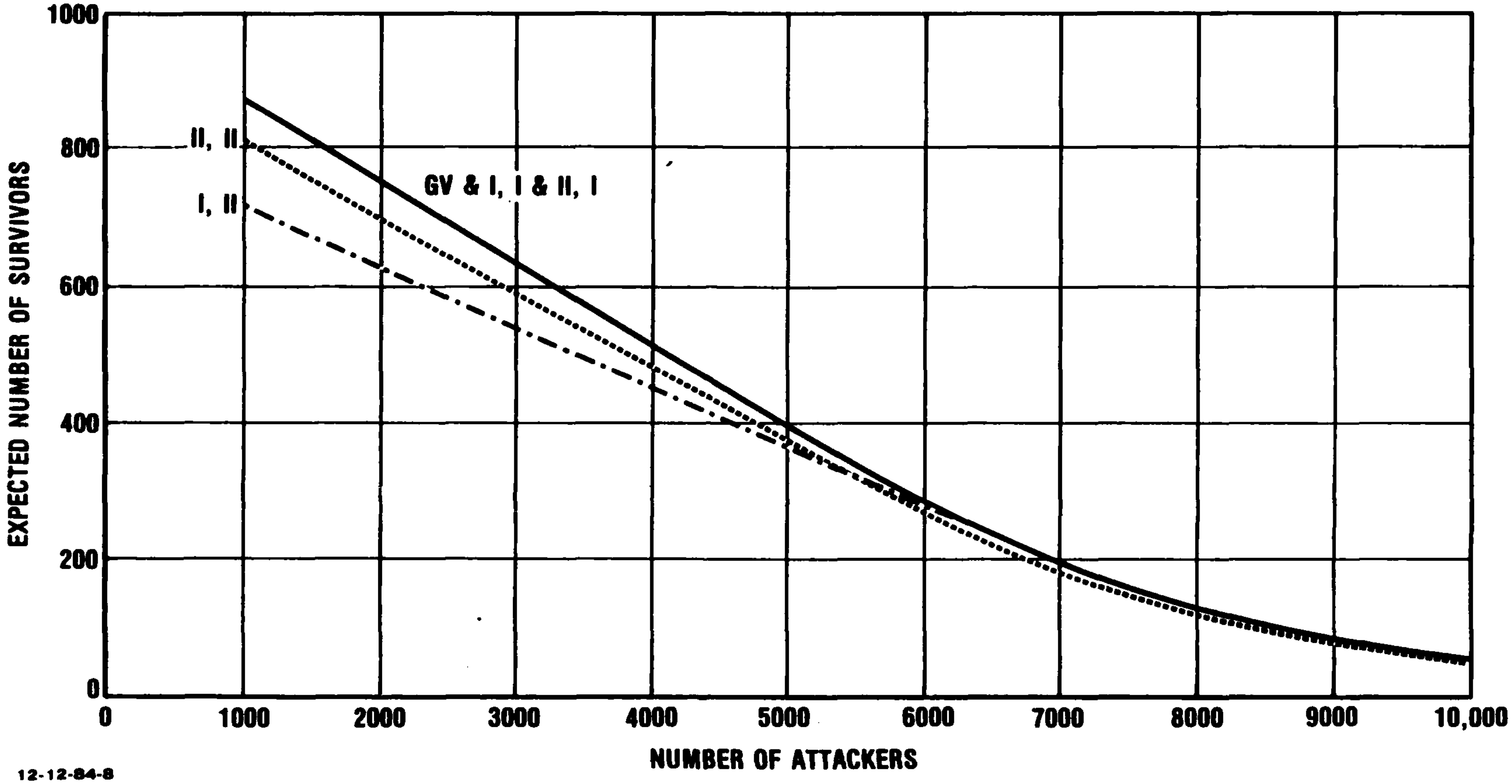

Figure 2. EXPECTED NUMBER OF SURVIVORS FOR a,d = .7, .7

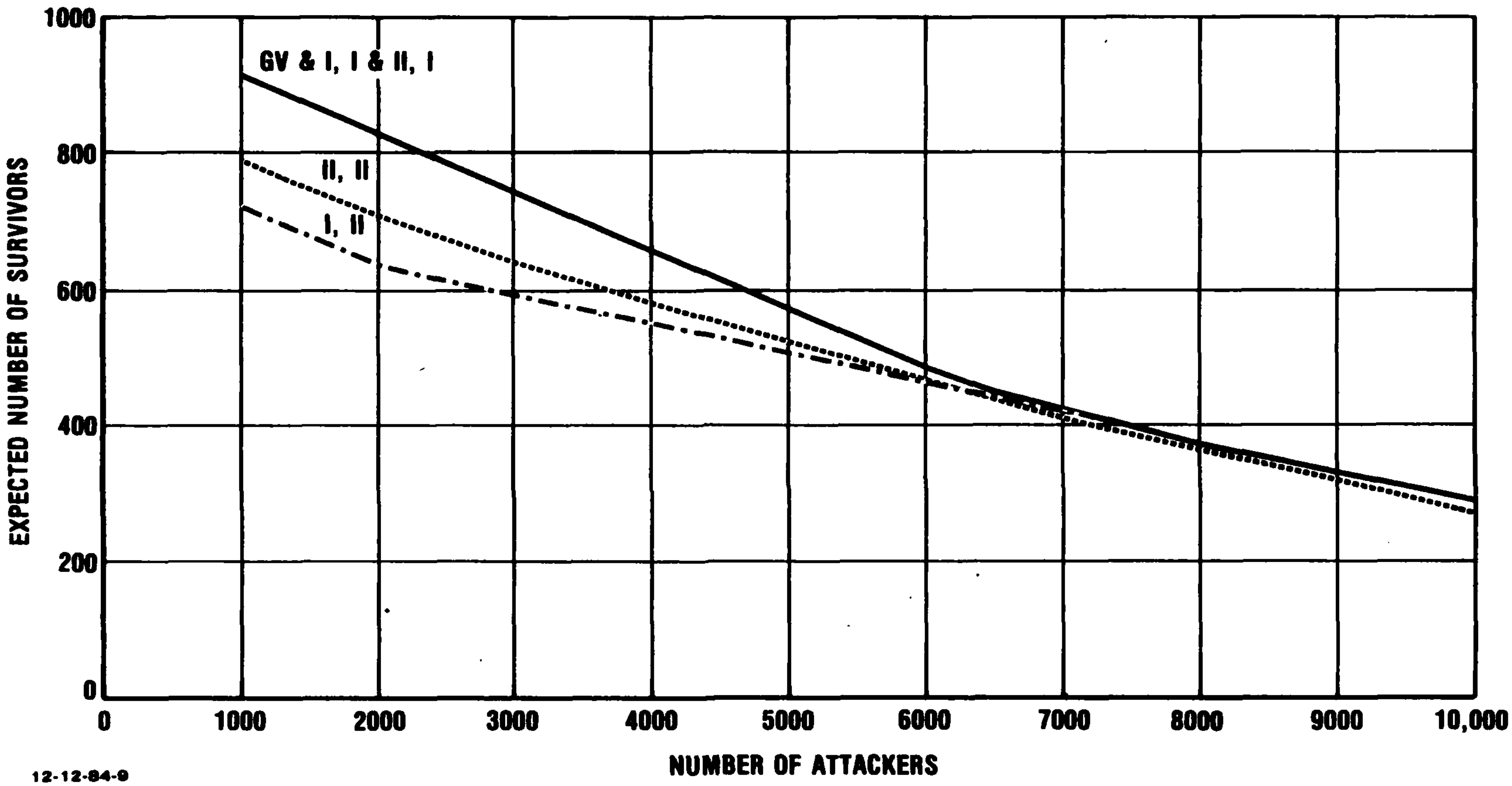

Figure 3. EXPECTED NUMBER OF SURVIVORS FOR a,d = .7, .9

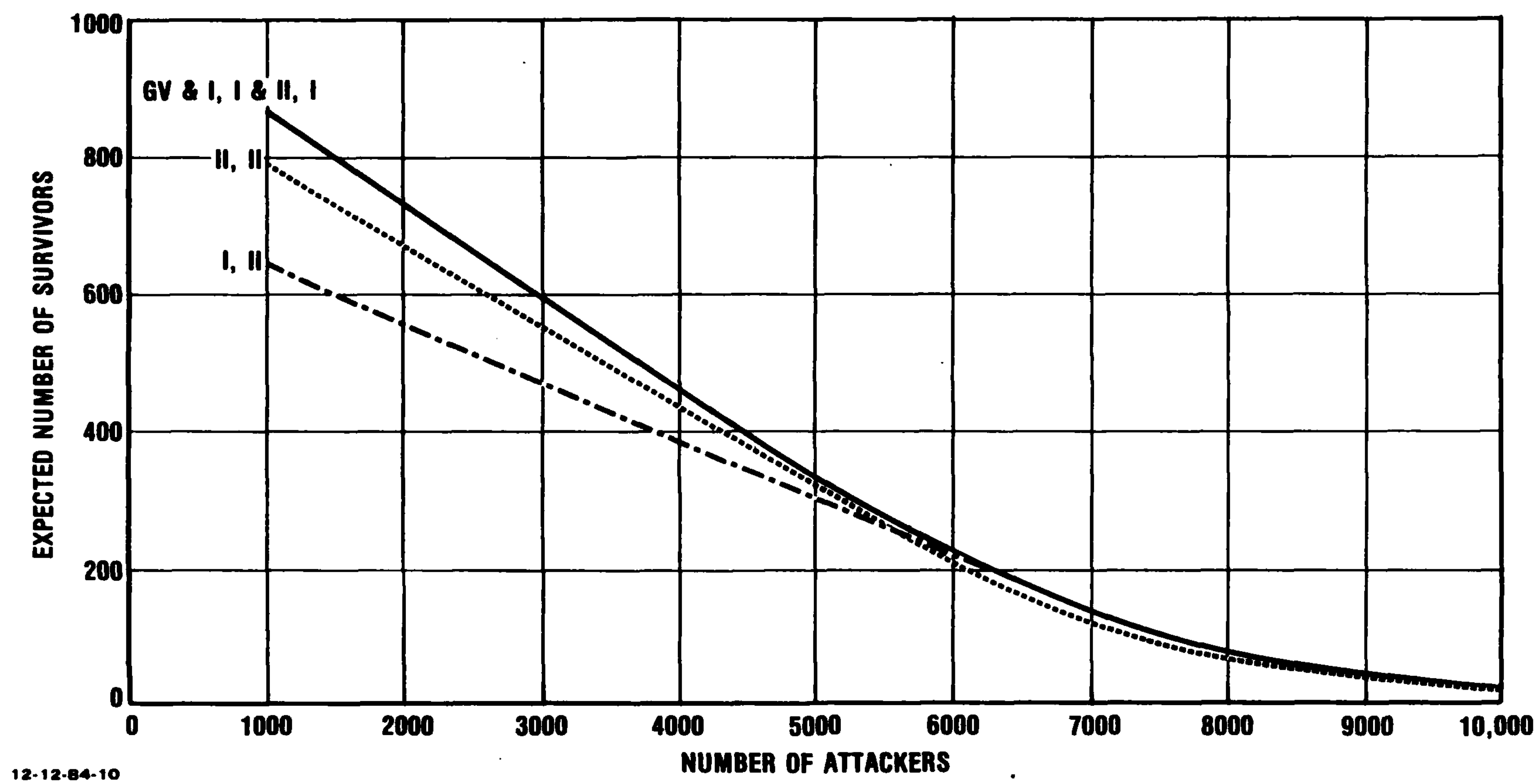

Figure 4. EXPECTED NUMBER OF SURVIVORS FOR a,d = .9, .7

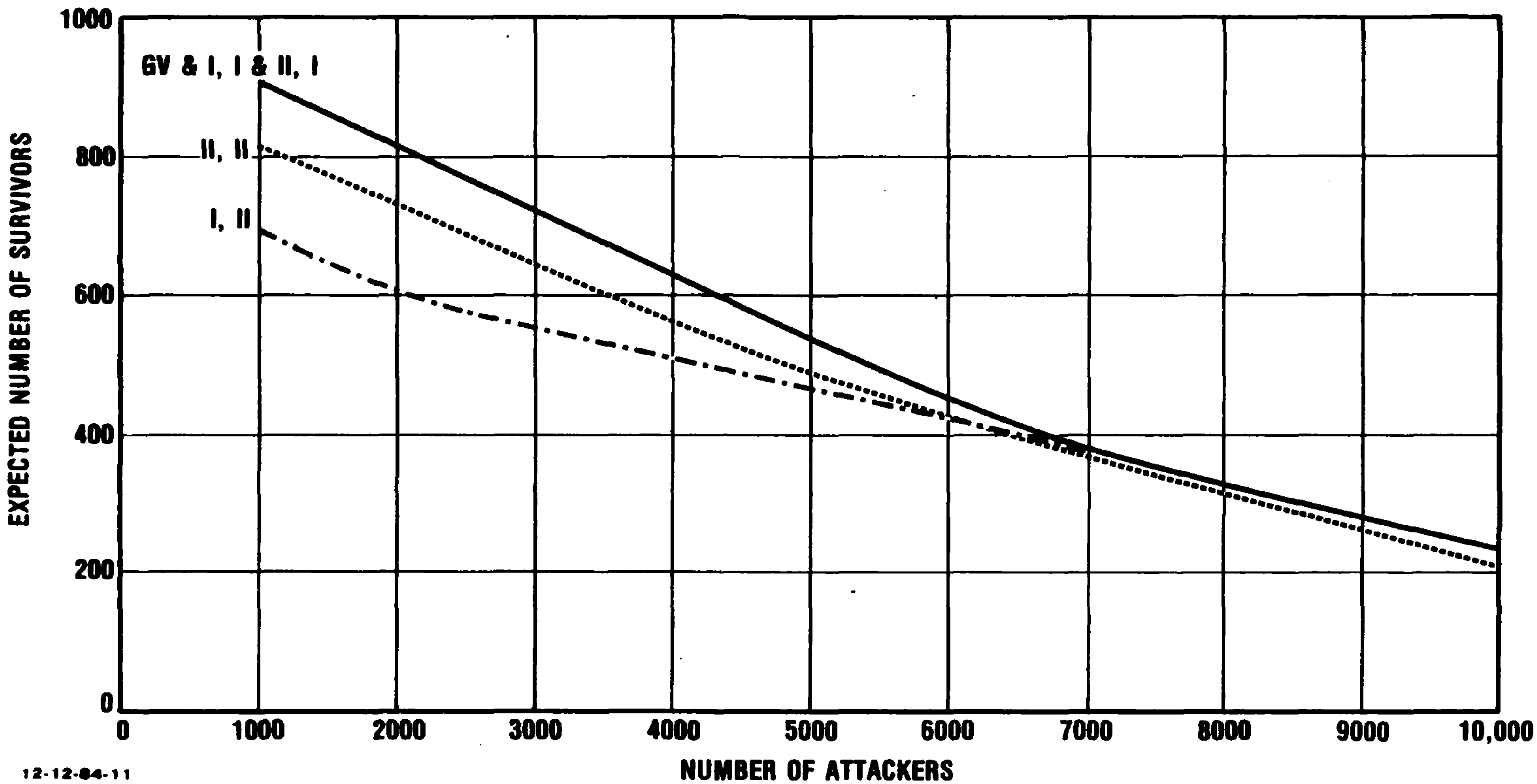

Figure 5. EXPECTED NUMBER OF SURVIVORS FOR a,d = .9, .9

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Thomas A. Brown and Christopher J. Hogg also furnished helpful review comments, for which we thank them.

Appendix A

SIMULTANEOUS ATTACK

SIMULTANEOUS ATTACK

Here we derive the particular expression adopted to compute the p_{ij}'s. We assume that a single target is under a simultaneous attack by A identical missiles, and is being defended by D identical interceptors. Let

- d = probability that a defending interceptor will destroy the attacking missile at which it is directed,
- a = probability that an attacking missile will destroy the target, given that it evades all defending interceptors.

We assume that the defense can see the entire attack, and must decide on the number of interceptors that it assigns to each of the attacking missiles.

It is easy to see that the probability of the target surviving an attack of n_j attacking missiles, each of which are being attacked by j defending interceptors, is

$$(1 - a(1 - d)^j)^{n_j}$$

so that the probability of the target surviving is

$$P(A,D) = \prod_{j=0}^{D} (1 - a(1 - d)^j)^{n_j} \quad . \qquad (2)$$

The defender wishes to maximize this.

We wish to show that the "uniform defense" obtained by spreading the D interceptors as equally as possible over the A attackers is optimal.

Consider any allocation of interceptors to attackers which is not uniform. Then there is a pair $i < j$ with n_i, $n_j > 0$ where $i + 2 \leq j$. Consider a new (and more uniform) allocation obtained by allocating $i + 1$ interceptors to one of the n_i attackers, and $j - 1$ interceptors to one of the n_j attackers. The probability that the target now survives is

$$\frac{(1 - a(1 - d)^{i+1})(1 - a(1 - d)^{j-1})}{(1 - a(1 - d)^{i})(1 - a(1 - d)^{j})}$$

times the old probability, and this is easily shown to be greater than 1.

Thus the most uniform of defenses assigns

$$i = [D/A] \text{ defenders to } n_i = A\,(1 - \langle D/A \rangle) \text{ attackers}$$

and

$$j = [D/A] + 1 \text{ defenders to } n_j = A\,\langle D/A \rangle \text{ attackers}$$

(where $[x]$ and $\langle x \rangle$ denote the integer and fractional parts of x).

Substituting these values into (2) yields

$$P(A,D) = (1-a(1-d)^{[D/A]+1})^{A\langle D/A \rangle}(1-a(1-d)^{[D/A]})^{A(1-\langle D/A \rangle)} .$$

Table 5 gives the numerical values of p_{ij} for the examples presented in Section V.

Table 5. VALUES of p_{ij} (SIMULTANEOUS ATTACK)

a=.7, d=.7

A \ D	0	1	2	3	4	5	6	7	8	9	10
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.3000	.7900	.9370	.9811	.9943	.9983	.9995	.9998	1.0000	1.0000	1.0000
2	.0900	.2370	.6241	.7402	.8780	.9193	.9626	.9755	.9887	.9926	.9966
3	.0270	.0711	.1872	.4930	.5848	.6936	.8227	.8614	.9019	.9444	.9571
4	.0081	.0213	.0562	.1479	.3895	.4620	.5479	.6499	.7708	.8071	.8451
5	.0024	.0064	.0169	.0444	.1169	.3077	.3650	.4329	.5134	.6090	.7223
6	.0007	.0019	.0051	.0133	.0351	.0923	.2431	.2883	.3420	.4056	.4811
7	.0002	.0006	.0015	.0040	.0105	.0277	.0729	.1920	.2278	.2702	.3204
8	.0001	.0002	.0005	.0012	.0032	.0083	.0219	.0576	.1517	.1799	.2134
9	.0000	.0001	.0001	.0004	.0009	.0025	.0066	.0173	.0455	.1199	.1422
10	.0000	.0000	.0000	.0001	.0003	.0007	.0020	.0052	.0137	.0360	.0947

7-10-86-1

a=.7, d=.9

A \ D	0	1	2	3	4	5	6	7	8	9	10
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.3000	.9300	.9930	.9993	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	.0900	.2790	.8649	.9235	.9860	.9923	.9986	.9992	.9999	.9999	1.0000
3	.0270	.0837	.2595	.8044	.8588	.9170	.9791	.9854	.9916	.9979	.9985
4	.0081	.0251	.0778	.2413	.7481	.7987	.8528	.9106	.9723	.9785	.9847
5	.0024	.0075	.0234	.0724	.2244	.6957	.7428	.7931	.8469	.9042	.9655
6	.0007	.0023	.0070	.0217	.0673	.2087	.6470	.6908	.7376	.7876	.8409
7	.0002	.0007	.0021	.0065	.0202	.0626	.1941	.6017	.6425	.6860	.7325
8	.0001	.0002	.0006	.0020	.0061	.0188	.0582	.1805	.5596	.5975	.6380
9	.0000	.0001	.0002	.0006	.0018	.0056	.0175	.0542	.1679	.5204	.5557
10	.0000	.0000	.0001	.0002	.0005	.0017	.0052	.0162	.0504	.1561	.4840

7-10-86-2

a=.9, d=.7

A \ D	0	1	2	3	4	5	6	7	8	9	10
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.1000	.7300	.9190	.9757	.9927	.9978	.9993	.9998	.9999	1.0000	1.0000
2	.0100	.0730	.5329	.6709	.8446	.8967	.9520	.9686	.9855	.9905	.9956
3	.0010	.0073	.0533	.3890	.4897	.6165	.7762	.8240	.8749	.9289	.9451
4	.0001	.0007	.0053	.0389	.2840	.3575	.4501	.5666	.7133	.7573	.8040
5	.0000	.0001	.0005	.0039	.0284	.2073	.2610	.3285	.4136	.5207	.6555
6	.0000	.0000	.0001	.0004	.0028	.0207	.1513	.1905	.2398	.3019	.3801
7	.0000	.0000	.0000	.0000	.0003	.0021	.0151	.1105	.1391	.1751	.2204
8	.0000	.0000	.0000	.0000	.0000	.0002	.0015	.0110	.0806	.1015	.1278
9	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0011	.0081	.0589	.0741
10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0008	.0059	.0430

7-10-86-3

a=.9, d=.9

A \ D	0	1	2	3	4	5	6	7	8	9	10
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.1000	.9100	.9910	.9991	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	.0100	.0910	.8281	.9018	.9821	.9901	.9982	.9990	.9998	.9999	1.0000
3	.0010	.0091	.0828	.7536	.8206	.8937	.9732	.9812	.9892	.9973	.9981
4	.0001	.0009	.0083	.0754	.6857	.7468	.8133	.8857	.9645	.9724	.9803
5	.0000	.0001	.0008	.0075	.0686	.6240	.6796	.7401	.8059	.8777	.9558
6	.0000	.0000	.0001	.0008	.0069	.0624	.5679	.6184	.6735	.7334	.7987
7	.0000	.0000	.0000	.0001	.0007	.0062	.0568	.5168	.5628	.6129	.6674
8	.0000	.0000	.0000	.0000	.0001	.0006	.0057	.0517	.4703	.5121	.5577
9	.0000	.0000	.0000	.0000	.0000	.0001	.0006	.0052	.0470	.4279	.4660
10	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0047	.0428	.3894

7-10-86-4

Appendix B

SEQUENTIAL ATTACK OF UNKNOWN SIZE

SEQUENTIAL ATTACK OF UNKNOWN SIZE

When the targets are under attack by an unknown number of attackers, the "Prim-Read" firing doctrine may be imposed at the targets (see, e.g., [illegible]). In this context, if there are D defenders at a target, we assume that d(1) are fired at the first attackers, d(2) at the second, and so on, where

$$\sum_{j=1}^{\infty} d(j) = D$$

and where the d(j)'s are selected so that

$$\max_{A=1,2,3\ldots} \left\{ \left(1 - \prod_{j=1}^{A} \left(1 - a(1-d)^{d(j)}\right) \right) /A \right\}$$

is minimized. Here a and d are defined as in Appendix A, and the quantity in brackets above is the probability that the target is destroyed per attacking weapon. In other words, the defense is set so that the attacker is (approximately) indifferent to the total number of RV's that he attacks with, in that their unit effectiveness is about the same.

With the firing doctrines set for each D=1,2,..., the values P(A,D) are computed from

$$P(A,D) = \prod_{j=1}^{A} \left(1 - a(1-d)^{d(j)}\right) \quad .$$

Table 6 lists these values for the single case a,d = .7, .9. Figure 6 exhibits the differences between the game values for the simultaneous and sequential cases.

The robust strategy y^{II} in the case of sequential attack of unknown size is

$$y^{II} = (0.245,\ 0.000,\ 0.034,\ 0.056,\ 0.068,\ 0.080,\ 0.020,\ 0.000,\ 0.000,\ 0.000,\ 0.497) \quad .$$

Table 6. VALUES OF p_{ij} (SEQUENTIAL ATTACK, SIZE UNKNOWN) FOR a,d = .7, .9

A \ D	0	1	2	3	4	5	6	7	8	9	10
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.3000	.9300	.9300	.9300	.9300	.9300	.9300	.9300	.9300	.9300	.9300
2	.0900	.2790	.8469	.8649	.8649	.8649	.8649	.8649	.8649	.8649	.8649
3	.0270	.0837	.2595	.8044	.8044	.8044	.8044	.8044	.8044	.8044	.8044
4	.0081	.0251	.0778	.2413	.7481	.7481	.7481	.7481	.7481	.7481	.7481
5	.0024	.0075	.0234	.0724	.2244	.6957	.6957	.6957	.6957	.6957	.6957
6	.0007	.0023	.0070	.0217	.0673	.2087	.6470	.6470	.6470	.6470	.6470
7	.0002	.0007	.0021	.0065	.0202	.0626	.1941	.6017	.6017	.6017	.6017
8	.0001	.0002	.0006	.0020	.0061	.0188	.0582	.1805	.5596	.5596	.5596
9	.0000	.0001	.0002	.0006	.0018	.0056	.0175	.0542	.1679	.5204	.5204
10	.0000	.0000	.0001	.0002	.0005	.0017	.0052	.0162	.0504	.1561	.4840

7-17-86-4

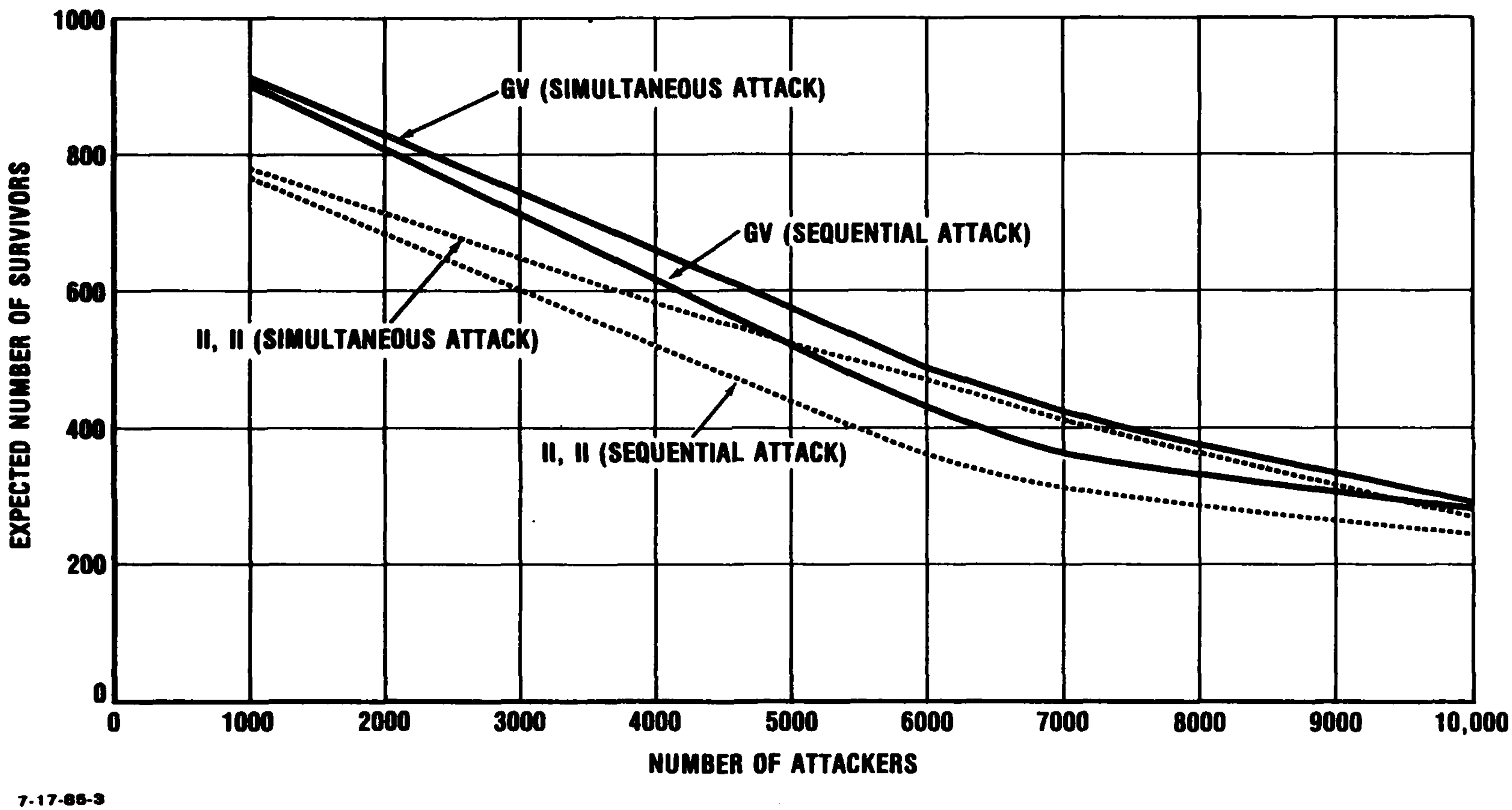

Figure 6. EXPECTED NUMBER OF SURVIVORS FOR SIMULTANEOUS AND SEQUENTIAL ATTACKS FOR a,d = .7, .9

Appendix C

SEQUENTIAL ATTACK OF KNOWN SIZE

SEQUENTIAL ATTACK OF KNOWN SIZE

Here we address the case where the attack is "sequential", i.e., there is enough time between successive attackers that they can be ordered and the attack size is known. We define, as before:

P(A,D) = probability that the target survives, given that it is under attack by A missiles and is optimally defended by D defenders.

Obviously, if the defender knows the value of A, he will defend uniformly according to the result of Appendix A. (A simultaneous attack can be considered sequential by numbering the attackers in any order.)

However, if the defender has a shoot-look-shoot capability, and sufficient time between arrivals, he can choose to structure his defense in volleys, with the prospect of saving defenders for use against future attackers.

Suppose the defender has time for two volleys against each incoming attacker. Let a be, as before, the kill probability of an attacking missile. Let

d = probability that a defending interceptor will destroy an attacking missile in the first volley

and

e = probability that a defending interceptor will destroy an attacking missile in the second volley.

Let

d(A) = number of interceptors to shoot at the first of A attacking missiles in the first volley

and

$e(A)$ = number of interceptors to shoot at the first of A attacking missiles in the second volley, given that the first volley has failed.

Then

$1-d(1-d)^{d(A)}$ is the probability that the first volley is successful,

$(1-d)^{d(A)}(1-(1-e)^{e(A)})$ is the probability that the first volley fails but the second is successful,

and

$(1-d)^{d(A)}(1-e)^{e(A)}(1-a)$ is the probability that both volleys fail and the attack also fails.

The following recursion holds:

$$P(A,D) = \max_{\substack{d(A),e(A)\,\varepsilon\, I^t \\ d(A)+e(A)\le D}} \left\{ (1-(1-d)^{d(A)})\ P(A-1,D-d(A)) + (1-d)^{d(A)}(1-a(1-e)^{e(A)})P(A-1,D-d(A)-e(A)) \right\}$$

with

$$P(0,D) = 1 \quad \text{for all } D\ \varepsilon\ I^+.$$

Given a, d and e, the recursion can be solved by dynamic programming to determine $P(A,D) = p_{ij}$. Note that the solution of this recursion would agree with the results of Appendix A in the case where e=0.

Obviously, the above recursion could be extended if the defender had more than two opportunities to protect himself.

Shoot-look-shoot capabilities in the Prim-Read context have also been investigated (Falk [2]).

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